Derivation of Mutual Coherence Function of the Pressure Field Envelope using the Parabolic Equation Method

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Introduction

This document outlines the steps needed to derive the Mutual Coherence Function (MCF) of the pressure field envelope, given a random sound speed profile, whose squared index of refraction exhibits Gaussian statistics in the vertical direction. Waveguide propagation is assumed, using a small angle Parabolic Equation method first proposed by Tappert[1]. With mathematical support from Novikov[2], Tatarskiĭ[3] derived both the mean and the second moment of the pressure field envelope. This was summarized by Barabanenkov[4] and later incorporated into Ishimaru's monograph[5]. Macaskill[6] extended Tatarskiĭ's work to incorporate arbitrary deterministic sound velocity profiles.

The Parabolic Equation

Following the derivation given by Tappert[1] and summarized by Jensen[7], we start with the Helmholtz equation in cylindrical coordinates,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2}{\partial z^2} + k_0^2 n^2 p = 0, \tag{1}$$

with p as the acoustic pressure, r is the range from the source (in meters), z is the depth in the water column (also in meters), k_0 is the reference acoustic wavenumber, $2\pi f/c_0$, and n is the index of refraction, $c_0/c(r,z)$. Acoustic pressure p can be separated into two parts; the pressure envelope, ψ, and the oscillatory part, expressed as a Hankel function,

$$p(r,z) = \psi(r,z)H_0^{(1)}(k_0 r). \tag{2}$$

The Hankel function satisfies the Bessel differential equation,

$$\frac{\partial^2 H_0^{(1)}(k_0 r)}{\partial r^2} + \frac{1}{r}\frac{\partial H_0^{(1)}(k_0 r)}{\partial r} + k_0^2 H_0^{(1)}(k_0 r) = 0 \tag{3}$$

whose solution can be expressed in asymptotic form for large values of $k_0 r$,

$$H_0^{(1)}(k_0 r) \simeq \sqrt{\frac{2}{\pi k_0 r}} e^{j(k_0 r - \pi/4)}. \tag{4}$$

Substitution of Equation 4 into Equation 1, and assuming $k_0 r \gg 1$ yields the simplified elliptic wave equation,

$$\frac{\partial^2 \psi}{\partial r^2} + j2k_0\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2-1)\psi = 0. \tag{5}$$

Finally, one applies a paraxial approximation,

$$\frac{\partial^2 \psi}{\partial r^2} \ll j2k_0\frac{\partial \psi}{\partial r} \tag{6}$$

which eliminates the first term of Equation 5. It restricts consideration to problems where the acoustic energy travels at angles $10 - 15^o$ from the horizontal axis, which is the case for the acoustic problems under study here. The result is the standard parabolic equation for underwater acoustics, as expressed by Tappert[1],

$$j2k_0\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2-1)\psi = 0. \tag{7}$$

Environmental Statistics

We assume the index of refraction is a function of range and depth, $n(r, z)$. The parabolic wave equation takes the index of refraction squared as an argument; we will use $\eta = n^2 - 1$ here. One can assume η is made up of two parts: a mean deterministic component, η_0, and a zero mean random component, η_1,

$$\eta(r, z) = \eta_0(r, z) + \eta_1(r, z). \tag{8}$$

The following conditions are assumed on the random component, η_1,

$$\begin{aligned} \langle \eta_1(r, z) \rangle &= 0 & (9) \\ \langle \eta_1(r, z)\eta_1(r', z') \rangle &= \delta(r - r')\Gamma_\eta(z - z') & (10) \\ \Gamma_\eta(z - z') &= \langle \eta_1(z)\eta_1(z') \rangle & (11) \end{aligned}$$

The zero mean condition (Equation 9) is assumed since η_0 incorporates any deterministic bias. The delta correlation (Equation 10) in the propagation direction, r, assumes the transverse (vertical) statistics dominate the stochastic nature of the waveguide, and that correlation in depth occurs only across the same range. η is assumed to be stationary in the wide sense (Equation 11); Γ_η depends only in the difference in depth between the two arguments, not on the absolute position in depth. While this does not accurately reflect realistic ocean conditions, it is suitable for the analytical results presented here.

Novikov's Functional Derivative Formula

Novikov[2] developed a method to solve for the correlation between Gaussian random functions and dependent functionals. One specific method dealt with functionals which were delta correlated in time and homogeneous in space. Consider the one dimensional case, with Gaussian random function $f_s(s)$, and its correlation $\langle f_i(s)f_k(s') \rangle = F_{ik}(s, s')$,

$$\langle f_s(s)R[f] \rangle = \int ds' \; F_{ik}(s, s') \left\langle \frac{\partial R[f]}{\partial f(s')ds'} \right\rangle \tag{12}$$

Novikov's objective was to prove this formula. The first step was to represent the function $R[f]$ as a functional Taylor series,

$$R[f] = R[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int ds_1 \cdots \int ds_n \; R^{(n)}_{i_1,\ldots,i_n}(s_1, \ldots, s_n) f_{i_1}(s_1) \cdots f_{i_n}(s_n) \tag{13}$$

where:

$$R^{(n)}_{i_1,\ldots,i_n}(s_1, \ldots, s_n) = \left. \frac{\partial^n R[f]}{\partial f_{i_1}(s_1)ds_1 \cdots f_{i_n}(s_n)ds_n} \right|_{f=0} \tag{14}$$

Multiplying by $f_i(s)$ and taking the average results in

$$\langle f_i(s)R[f] \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \int ds_1 \cdots \int ds_n \; R^{(n)}_{i_1,\ldots,i_n}(s_1, \ldots, s_n) \langle f_i(s) f_{i_1}(s_1) \cdots f_{i_n}(s_n) \rangle . \tag{15}$$

The product of an even number of jointly Gaussian random variables can be expressed as a sum of the the mean values of all possible pairwise combinations[8]. The product of an odd number of jointly Gaussian random variables is zero. For example, given a jointly Gaussian set of zero mean random variables $x_1, \ldots, x_N$, with covariances $\lambda_{ik} = \langle x_i x_k \rangle, 1 \leq i, j \leq N$, then for *any* set of integers $i_1, i_2, \ldots, i_L$,

$$\langle x_{i_1} x_{i_2} \cdots x_{i_L} \rangle = \begin{cases} 0 & L \text{ odd} \\ \sum \lambda_{k_1 k_2} \lambda_{k_3 k_4} \cdots \lambda_{k_{L-1} k_L} & L \text{ even} \end{cases} \tag{16}$$

with the summation over all distinct pairings of $i_1, i_2, \ldots, i_L$. This can be applied to the right hand side of Equation 15 and rewritten as

$$\langle f_i(s) f_{i_1}(s_1)\cdots f_{i_n}(s_n)\rangle = \sum_{\alpha=1}^{n} \langle f_i(s) f_{i_\alpha}(s_\alpha)\rangle \left\langle f_{i_1}(s_1)\cdots f_{i_{\alpha-1}}(s_{\alpha-1}) f_{i_{\alpha+1}}(s_{\alpha+1})\cdots f_{i_n}(s_n)\right\rangle . \tag{17}$$

Assuming $F_{i1}(s, s_1) = F_{i2}(s, s_2) = \cdots = F_{in}(s, s_n)$, which can be inferred from the jointly Gaussian nature of $f_i, f_{i1}, \ldots, f_{in}$, then Equation 17, can be substituted into Equation 15 to yield

$$\begin{aligned} \langle f_i(s) R[f]\rangle = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_1\ F_{ii_1}(s, s_1) \\ \times \left[\int ds_2 \cdots \int ds_n\ R^{(n)}_{i_1,i_2,\ldots,i_n}(s_1, s_2, \ldots, s_n) \langle f_{i_2}(s)\cdots f_{i_n}(s_n)\rangle\right]. \end{aligned} \tag{18}$$

In a parallel approach, the functional derivative of the Taylor series, Equation 13 can be taken,

$$\frac{\partial R[f]}{\partial f_k(s')ds'} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_2 \int ds_n\ R^{(n)}_{k,i_2,\ldots,i_n}(s', s_2, \ldots, s_n) f_{i_2}\cdots f_{i_n}(s_n). \tag{19}$$

Upon taking the average value of Equation 19 and substituting it into Equation 12, one obtains

$$\langle f_k(s) R[f]\rangle = \int ds'\ F_{ik}(s, s') \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_2 \int ds_n\ R^{(n)}_{k,i_2,\ldots,i_n}(s', s_2, \ldots, s_n) \langle f_{i_2}\cdots f_{i_n}(s_n)\rangle \tag{20}$$

Note Equations 18 and 20 are equivalent; this shows Equation 12 to hold true.

Mean Pressure Field Envelope

In solving for the second moment (correlation) statistics, of the pressure field envelope ψ, one can start by solving for the first order (mean) statistics. Starting with the small angle, two dimensional Parabolic Wave Equation[1], and assuming $\eta(r,z)$ is zero mean:

$$j2k_0 \frac{\partial\psi(r,z)}{\partial r} + \frac{\partial^2\psi(r,z)}{\partial z^2} + k_0^2 \eta_1(r,z)\psi(r,z) = 0, \tag{21}$$

one can take the average of the expression, substituting $\bar{\psi}(r,z) = \langle\psi(r,z)\rangle$ and, in this two-dimensional case, the transverse Laplacian $\nabla_t^2\bar{\psi}(r,z) = \partial^2\bar{\psi}(r,z)/\partial z^2$.

$$j2k_0 \frac{\partial\bar{\psi}(r,z)}{\partial r} + \nabla_t^2\bar{\psi}(r,z) + k_0^2 \underbrace{\langle\eta_1(r,z)\psi(r,z)\rangle}_{g_1(r,z)} = 0 \tag{22}$$

We must now solve for $g_1(r,z)$. Recall Novikov's functional derivative formula[2]:

$$\langle\eta_1(r,z) Z[\eta_1]\rangle = \int dr' \int dz'\ \langle\eta_1(r,z)\eta_1(r',z')\rangle \left\langle \frac{\partial Z[\eta_1]}{\partial\eta_1(r',z')}\right\rangle \tag{23}$$

One can substitute Equations 10, 11 and $Z[\eta_1] = \psi$ to obtain

$$g_1(r,z) = \int dz'\ \Gamma_\eta(z - z') \left\langle \frac{\partial\psi(r,z)}{\partial\eta_1(r,z')}\right\rangle . \tag{24}$$

One must solve for the average functional derivative, $\left\langle \frac{\partial\psi(r,z)}{\partial\eta_1(r,z')}\right\rangle$. One can start again with the small angle Parabolic Equation 21, and integrate with respect to r,

$$j2k_0\psi(r,z) - j2k_0\psi(0,z) + \nabla_t^2 \int_0^r \psi(\xi,z)d\xi + k_0^2 \int_0^r \eta_1(\xi,z)\psi(\xi,z)d\xi = 0 \tag{25}$$

One can introduce the integrated delta function,

$$\Theta(\xi) = \int_{-\infty}^{\xi} \delta(\xi')d\xi' = \begin{cases} 0 & \text{for} \quad \xi < 0 \\ 1/2 & \text{for} \quad \xi = 0 \\ 1 & \text{for} \quad \xi > 0 \end{cases} \tag{26}$$

and insert it into the last term of Equation 25,

$$\begin{aligned} j2k_0\psi(r,z) - j2k_0\psi(0,z) + \nabla_t^2 \int_0^r \psi(\xi,z)d\xi \\ +k_0^2 \int_0^\infty d\xi \int dz''\Theta(r-xi)\delta(z-z'')\eta_1(\xi,z'')\psi(\xi,z'') = 0. \end{aligned} \tag{27}$$

One then takes the functional derivative, $\partial/\partial\eta_1(r',z')$, assuming the functional identity,

$$\frac{\partial}{\partial\eta_1(r',z')}\eta_1(\xi,z'') = \delta(\xi-r')\delta(z''-z'), \tag{28}$$

to obtain

$$\begin{aligned} j2k_0\frac{\partial\psi(r,z)}{\partial\eta_1(r',z')} + \nabla_t^2 \int_0^r \frac{\partial\psi(\xi,z)}{\partial\eta_1(r',z')}d\xi \\ +k_0^2\Theta(r-r')\delta(z-z')\psi(r',z') + k_0^2 \int_0^\infty d\xi \int dz''\, \Theta(r-\xi)\delta(z-z'')\eta_1(\xi,z'')\frac{\partial\psi(\xi,z'')}{\partial\eta_1(r',z')} = 0. \end{aligned} \tag{29}$$

One condition of the small-angle approximation assumes there is no backscatter (energy reflecting back toward the source). With this in mind, one can assume that the values of $\psi(\xi,z)$ depend only on the index of refraction, $\eta(r',z')$ when $r' < \xi$. Consequently, the functional derivative of the pressure field envelope,

$$\frac{\partial\psi(\xi,z)}{\partial\eta_1(r',z)} = 0 \quad \text{when } r' > \xi. \tag{30}$$

This allows one to set the lower limit of integration in Equation 29 to r'. Integrating through (taking care to change the upper limit of integration in the last term, in accordance with Equation 26), one obtains

$$\begin{aligned} j2k_0\frac{\partial\psi(r,z)}{\partial\eta_1(r',z')} + \nabla_t^2 \int_{r'}^r \frac{\partial\psi(\xi,z)}{\partial\eta_1(r',z')}d\xi \\ +k_0^2\Theta(r-r')\delta(z-z')\psi(r',z') + k_0^2 \int_{r'}^r d\xi\, \eta_1(\xi,z)\frac{\partial\psi(\xi,z)}{\partial\eta_1(r',z')} = 0. \end{aligned} \tag{31}$$

As r' approaches r, the integral terms vanish, resulting in

$$j2k_0\frac{\partial\psi(r,z)}{\partial\eta_1(r,z')} + \frac{k_0^2}{2}\delta(z-z')\psi(r,z') = 0, \tag{32}$$

which can be rewritten and conjugated as

$$\frac{\partial\psi(r,z)}{\partial\eta_1(r,z')} = \frac{jk_0}{4}\delta(z-z')\psi(r,z) \tag{33}$$

$$\frac{\partial\psi^*(r,z)}{\partial\eta_1(r,z')} = -\frac{jk_0}{4}\delta(z-z')\psi^*(r,z). \tag{34}$$

This can be substituted into Equation 24 to find

$$\begin{aligned} g_1(r,z) &= \int dz' \Gamma_\eta(z-z') \left\langle \frac{jk_0}{4}\delta(z-z')\psi(r,z)\right\rangle \\ &= \Gamma_\eta(0)\frac{jk_0}{4}\bar{\psi}(r,z), \end{aligned} \tag{35}$$

and finally arrive at the differential equation for the propagation of the mean acoustic field envelope,

$$j2k_0\frac{\partial\bar{\psi}(r,z)}{\partial r} + \nabla_t^2\bar{\psi}(r,z) + \frac{jk_0^3}{4}\Gamma_\eta(0)\bar{\psi}(r,z) = 0. \tag{36}$$

Mutual Coherence Function

A similar approach can be used to solve for the transverse mutual coherence function,

$$\Gamma_\psi(r,z_1,z_2) = \langle \psi(r,z_1)\psi^*(r,z_2)\rangle\,. \tag{37}$$

We start with the small angle, two-dimensional Parabolic Equation, Equation 21, and substitute z_1 for z,

$$j2k_0\frac{\partial\psi(r,z_1)}{\partial r} + \frac{\partial^2\psi(r,z_1)}{\partial z_1^2} + k_0^2\eta_1(r,z_1)\psi(r,z_1) = 0, \tag{38}$$

and multiply it by $\psi^*(r,z_2)$,

$$j2k_0\frac{\partial\psi(r,z_1)}{\partial r}\psi^*(r,z_2) + \frac{\partial^2\psi(r,z_1)}{\partial z_1^2}\psi^*(r,z_2) + k_0^2\eta_1(r,z_1)\psi(r,z_1)\psi^*(r,z_2) = 0, \tag{39}$$

Substituting $\psi(r,z_2)$ for $\psi(r,z_1)$ in Equation 38, taking the conjugate of the equation, and multiplying the result by $\psi(r,z_1)$, one finds

$$-j2k_0\frac{\partial\psi^*(r,z_2)}{\partial r}\psi(r,z_1) + \frac{\partial^2\psi^*(r,z_2)}{\partial z_2^2}\psi(r,z_1) + k_0^2\eta_1(r,z_2)\psi(r,z_1)\psi^*(r,z_2) = 0. \tag{40}$$

Subtracting Equation 40 from Equation 39, and taking the average,

$$j2k_0\frac{\partial\Gamma_\psi}{\partial r} + \left(\nabla_{t1}^2 - \nabla_{t2}^2\right)\Gamma_\psi + k_0^2 g_2(r,z_1,z_2) = 0 \tag{41}$$

$$\text{where:}\quad \frac{\partial\Gamma_\psi}{\partial r} = \left\langle \frac{\partial\psi(r,z_1)}{\partial r}\psi^*(r,z_2) + \psi(r,z_1)\frac{\partial\psi^*(r,z_2)}{\partial r}\right\rangle \tag{42}$$

$$\nabla_{tN}^2\Gamma_\psi = \frac{\partial^2}{\partial z_N^2}\langle\psi(r,z_1)\psi^*(r,z_2)\rangle \tag{43}$$

$$g_2(r,z_1,z_2) = \langle[\eta_1(r,z_1) - \eta_1(r,z_2)]\,\psi(r,z_1)\psi^*(r,z_2)\rangle \tag{44}$$

We must now evaluate g_2. Substituting $Z(r,z_1,z_2) = \psi(r,z_1)\psi^*(r,z_2)$, into Equation 23, one obtains the equation pair,

$$\langle\eta_1(r,z_1)Z(r,z_1,z_2)\rangle = \int dr' \int dz_1' \,\langle\eta_1(r,z_1)\eta_1(r',z_1')\rangle \left\langle\frac{\partial Z(r,z_1,z_2)}{\partial\eta_1(r',z_1')}\right\rangle \tag{45}$$

$$\langle\eta_1(r,z_2)Z(r,z_1,z_2)\rangle = \int dr' \int dz_2' \,\langle\eta_1(r,z_2)\eta_1(r',z_2')\rangle \left\langle\frac{\partial Z(r,z_1,z_2)}{\partial\eta_1(r',z_2')}\right\rangle. \tag{46}$$

Using the directional delta assumption of Equation 10, these can be rewritten as

$$\langle \eta_1(r,z_1) Z(r,z_1,z_2) \rangle = \int dz_1' \, \Gamma_\eta(z_1 - z_1') \left\langle \frac{\partial Z(r,z_1,z_2)}{\partial \eta_1(r,z_1')} \right\rangle \tag{47}$$

$$\langle \eta_1(r,z_2) Z(r,z_1,z_2) \rangle = \int dz_2' \, \Gamma_\eta(z_2 - z_2') \left\langle \frac{\partial Z(r,z_1,z_2)}{\partial \eta_1(r,z_2')} \right\rangle . \tag{48}$$

with

$$\frac{\partial Z(r,z_1,z_2)}{\partial \eta_1(r,z_1')} = \frac{\partial \psi(r,z_1)}{\partial \eta_1(r,z_1')} \psi^*(r,z_2) + \psi(r,z_1) \frac{\partial \psi^*(r,z_2)}{\partial \eta_1(r,z_1')} . \tag{49}$$

Application of Equations 33 and 34 results in

$$\frac{\partial Z(r,z_1,z_2)}{\partial \eta_1(r,z_1')} = \frac{jk_0}{4} \delta(z_1 - z_1') \psi(r,z_1) \psi^*(r,z_2) - \frac{jk_0}{4} \delta(z_2 - z_1') \psi(r,z_1) \psi^*(r,z_2). \tag{50}$$

Combining terms and taking the average, one obtains

$$\left\langle \frac{\partial Z(r,z_1,z_2)}{\partial \eta_1(r,z_1')} \right\rangle = \frac{jk_0}{4} \Gamma_\psi(r,z_1,z_2) \left[\delta(z_1 - z_1') - \delta(z_2 - z_1') \right] . \tag{51}$$

Substitution of this expression into Equation 47 and evaluating the integral yields

$$\langle \eta_1(r,z_1) Z(r,z_1,z_2) \rangle = \frac{jk_0}{4} \left[\Gamma_\eta(0) - \Gamma_\eta(z_1 - z_2) \right] \Gamma_\psi(r,z_1,z_2). \tag{52}$$

In a similar manner, the functional with respect to $\eta(r, z_2')$ can be evaluated and substituted into Equation 48 to obtain

$$\langle \eta_1(r,z_2) Z(r,z_1,z_2) \rangle = \frac{jk_0}{4} \left[\Gamma_\eta(z_1 - z_2) - \Gamma_\eta(0) \right] \Gamma_\psi(r,z_1,z_2). \tag{53}$$

Subtracting Equation 53 from Equation 52 yields an expression for $g_2(r, z_1, z_2)$,

$$g_2(r,z_1,z_2) = \langle \left[\eta_1(r,z_1) - \eta_1(r,z_2) \right] \psi(r,z_1) \psi^*(r,z_2) \rangle = \frac{jk_0}{2} \left[\Gamma_\eta(0) - \Gamma_\eta(z_1 - z_2) \right] \Gamma_\psi(r,z_1,z_2) \tag{54}$$

which, when substituted into Equation 41 results in the parabolic equation for the Mutual Coherence Function of the pressure field envelope, ψ,

$$\left\{ j2k_0 \frac{\partial}{\partial r} + \left(\nabla_{t1}^2 - \nabla_{t2}^2 \right) + \frac{jk_0^3}{2} \left[\Gamma_\eta(0) - \Gamma_\eta(z_1 - z_2) \right] \right\} \Gamma_\psi(r,z_1,z_2) = 0. \tag{55}$$

Mutual Coherence Function For Arbitrary Refractive Indices

Equations 21, 36, and 55 all assume $\eta(r, z)$ is a zero mean Gaussian random variable,

$$\eta(r,z) = \eta_0(r,z) + \eta_1(r,z) \tag{56}$$

where $\eta_0(r, z)$ is zero for all values. Such a simplification does not allow one to consider realistic, depth dependent sound velocity profiles or their associated indices of refraction. Modification of the equations of propagation to allow for nonzero η_0 is relatively straightforward; the result was summarized by Macaskill and Uscinski[6].

The small angle parabolic wave equation (21) can be modified by replacing $\eta_1(r, z)$ with $\eta(r, z)$ and expanding the result,

$$j2k_0 \frac{\partial \psi(r,z)}{\partial r} + \frac{\partial^2 \psi(r,z)}{\partial z^2} + k_0^2 \eta_0(r,z) \psi(r,z) + k_0^2 \eta_1(r,z) \psi(r,z) = 0. \tag{57}$$

A similar approach can be taken with mean pressure field envelope equation (36),

$$\begin{aligned} j2k_0 \frac{\partial \bar{\psi}(r,z)}{\partial r} + \nabla_t^2 \bar{\psi}(r,z) + k_0^2 \left\langle \left[\eta_0(r,z) + \eta_1(r,z)\right] \psi(r,z) \right\rangle &= 0 \\ j2k_0 \frac{\partial \bar{\psi}(r,z)}{\partial r} + \nabla_t^2 \bar{\psi}(r,z) + k_0^2 \eta_0(r,z) \left\langle \psi(r,z) \right\rangle + k_0^2 \left\langle \eta_1(r,z) \psi(r,z) \right\rangle &= 0 \\ j2k_0 \frac{\partial \bar{\psi}(r,z)}{\partial r} + \nabla_t^2 \bar{\psi}(r,z) + k_0^2 \eta_0(r,z) \bar{\psi}(r,z) + \frac{jk_0^3}{4} \Gamma_\eta(0) \bar{\psi}(r,z) &= 0. \end{aligned} \tag{58}$$

For the Mutual Coherence Function, one can substitute $\eta_0(r,z)$ into the expression for $g_2(r,z_1,z_2)$, (Equation 44), resulting in

$$\begin{aligned} g_2'(r,z_1,z_2) &= \left\langle \left\{ \left[\eta_0(r,z_1) + \eta_1(r,z_1)\right] - \left[\eta_0(r,z_2) + \eta_1(r,z_2)\right] \right\} \psi(r,z_1) \psi^*(r,z_2) \right\rangle \\ &= \left\langle \left\{ \left[\eta_0(r,z_1) - \eta_0(r,z_2)\right] + \left[\eta_1(r,z_1) - \eta_1(r,z_2)\right] \right\} \psi(r,z_1) \psi^*(r,z_2) \right\rangle \\ &= \left[\eta_0(r,z_1) - \eta_0(r,z_2)\right] \Gamma_\psi(r,z_1,z_2) + \left\langle \left[\eta_1(r,z_1) - \eta_1(r,z_2)\right] \psi(r,z_1) \psi^*(r,z_2) \right\rangle \end{aligned} \tag{59}$$

Finally, placing Equation 54 into the last term of Equation 59 yields the modified MCF for an arbitrary deterministic refractive index,

$$\left\{ j2k_0 \frac{\partial}{\partial r} + \left(\nabla_{t1}^2 - \nabla_{t2}^2\right) + k_0^2 \left[\eta_0(r,z_1) - \eta_0(r,z_2)\right] + \frac{jk_0^3}{2} \left[\Gamma_\eta(0) - \Gamma_\eta(z_1 - z_2)\right] \right\} \Gamma_\psi(r,z_1,z_2) = 0. \tag{60}$$

References

[1] J. B. Keller and J. S. Papadakis, eds., *The Parabolic Approximation Method*, vol. 70 of *Lecture Notes in Physics*, ch. 5, pp. 224–280. Berlin: Springer-Verlag, 1977.

[2] E. A. Novikov, "Functionals and the random-force method in turbulence theory," *Soviet Physics JETP*, vol. 20, pp. 1290–1294, May 1965.

[3] V. I. Tatarskiĭ, "Light propagation in a medium with random refractive index inhomogeneities in the markov random process approximation," *Soviet Physics JETP*, vol. 29, pp. 1133–1138, December 1969.

[4] Y. N. Barabanenkov, Y. A. Kravtsov, S. M. Rytov, and V. I. Tamarskiĭ, "Status of the theory of propagation of waves in a randomly inhomogeneous medium," *Soviet Physics Uspekhi*, vol. 13, no. 5, pp. 551–575, 1971.

[5] A. Ishimaru, *Wave Propagationm and Scattering in Random Media*. IEEE/OUP Series on Electromagnetic Wave Theory, IEEE Press, 1997.

[6] C. C. Macaskill and B. J. Uscinski, "Propagation in waveguides containing random irregularities: the second moment equation," *Proceedings of the Royal Society of London, Series A*, vol. 377, pp. 73–98, 1981.

[7] F. B. Jensen, W. A. Kuperman, M. B. Porter, and H. Schmidt, *Computational Ocean Acoustics*. New York: American Institute of Physics Press, 1994.

[8] A. S. Willsky, G. Shapiro, and G. W. Wornell, *6.432 Stochastic Processes, Detection, and Estimation: Supplementary Notes*. Cambridge, Massachusetts: MIT Department of Electrical Engineering and Computer Science, 1995.