Derived Distributions of $\mathbf{p(r,z)}$ using the Finite Difference Implementation of the Parabolic Wave Equation

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January 6, 2000

Introduction

The pressure field generated from an acoustic source is dependent on several parameters: the geometry of the water column, source, and receiver, as well as the environmental parameters of the propagation medium. Included in these are the bathymetry, sediment layer density, and the speed of sound in the water column.

Given a random sound speed profile, $c(z)$, and its probabilistic distribution, $p_c(c)$, one could calculate the probabilistic distribution of the pressure field $p_p(p)$ at the receiver. This paper takes one through the steps of this calculation, giving an analytical result. A finite difference parabolic equation (FD-PE) approach is taken to calculating the acoustic pressure field; the probabilistic distribution calculated here would be valid only in situations where the FD-PE model is valid.

Background

We start with the derivation provided in Chapter 6 of Computational Ocean Acoustics, by Jensen et al. This, in turn, recounts the derivation made by Tappert in 1977. We start with the Helmholtz equation,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0, \tag{1}$$

where $p(r,z)$ is the acoustic pressure evaluated at range r and depth z, $k_0 = \omega/c_0$ is a reference wavenumber, and $n(r,z) = c_0/c(r,z)$ is the index of refraction.

One candidate solution for this equation would be the Hankel function $H_0^{(1)}(k_0 r)$ multiplied by an envelope function, $\psi(r,z)$,

$$p(r,z) = \psi(r,z) H_0^{(1)}(k_0 r) \tag{2}$$

The Hankel function satisfies the Bessel differential equation,

$$\frac{\partial^2 H_0^{(1)}(k_0 r)}{\partial r^2} + \frac{1}{r}\frac{\partial H_0^{(1)}(k_0 r)}{\partial r} + k_0^2 H_0^{(1)}(k_0 r) = 0, \tag{3}$$

and its asymptotic form can be substituted for large $k_0 r$,

$$H_0^{(1)}(k_0 r) \simeq \sqrt{\frac{2}{\pi k_0 r}} e^{j(k_0 r - \frac{\pi}{r})}. \tag{4}$$

Substitution of the trial solution (Equation 2) into the Helmholtz equation (Equation 1) gives

$$\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{2}{H_0^{(1)}(k_0 r)} \frac{\partial H_0^{(1)}(k_0 r)}{\partial r} + \frac{1}{r} \right) \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 (n^2 - 1)\psi = 0. \tag{5}$$

Next, substitute the asymptotic Hankel function solution (Equation 4) to yield

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 (n^2 - 1)\psi = 0. \tag{6}$$

The final step in Tappert's derivation is to use a small angle paraxial approximation, which assumes

$$\frac{\partial^2 \psi}{\partial r^2} \ll 2ik_0 \frac{\partial \psi}{\partial r}, \tag{7}$$

and allows one to drop the $\frac{\partial^2 \psi}{\partial r^2}$ term to arrive at the standard parabolic wave equation,

$$2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 (n^2 - 1)\psi = 0. \tag{8}$$

This solution lends itself to straightforward analytic expressions for simple environments. Unfortunately, the small angle approximation limits the effectiveness of the parabolic wave equation to pressure fields which propagate in a sector 10-15 degrees from the horizontal axis. For long range, deep water acoustics, most low frequency acoustic the energy propagates in the SOFAR channel, which is within this sector. Unfortunately, shorter range, shallow water acoustic propagation results in significant energy distributed outside of the small angle sector.

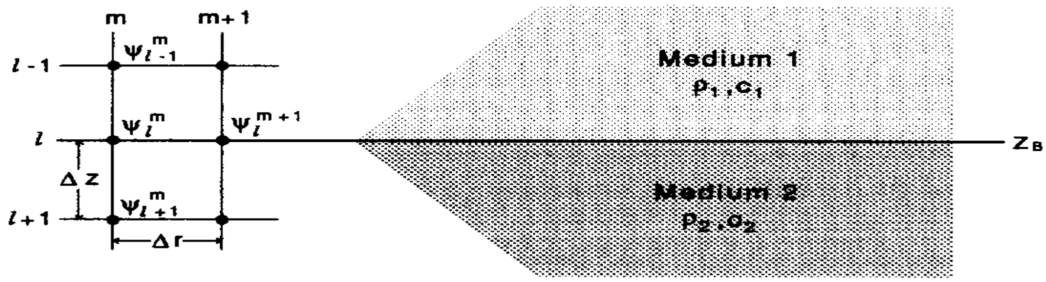

Figure 1: Plot of finite difference cell (from Jensen et al.)

For complicated environments which do not lend themselves to simple analytical expressions, a finite element or finite difference approach may be warranted. One divides the water column up into a (r, z) grid, iteratively solving for the pressure field in range and depth. By making the grid spacing $(\Delta r, \Delta z)$ small, one can calculate accurate solutions to the parabolic wave equation (see Figure 1).

With the finite difference approach, one need not make the small angle assumption used in Equation 8. Starting again with the far field Helmholtz Equation,

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2(n^2 - 1)\psi + \frac{\partial^2 \psi}{\partial z^2} = 0, \tag{9}$$

and applying the boundary conditions at an arbitrary grid boundary point z_B that (1) the pressure, and (2) the vertical derivative of the fields will be equal at this boundary point,

$$\psi_1(r, z_B) = \psi_2(r, z_B) \tag{10}$$

$$\left.\frac{1}{\rho_1}\frac{\partial \psi_1}{\partial z}\right|_{z=z_B} = \left.\frac{1}{\rho_1}\frac{\partial \psi_2}{\partial z}\right|_{z=z_B} \tag{11}$$

For the first medium (1), one can perform a Taylor series expansion of ψ_{l-1}^m around ψ_l^m,

$$\psi_{l-1}^m = \psi_l^m - \Delta z \frac{\partial \psi_l^m}{\partial z} + \frac{(\Delta z)^2}{2}\frac{\partial^2 \psi_l^m}{\partial z^2} + \cdots \tag{12}$$

Solving for the second derivative of ψ with respect to z,

$$\frac{\partial^2 \psi_1}{\partial z^2} = -\frac{2}{(\Delta z)^2}\left(\psi_1 - \psi_{l-1}^m\right) + \frac{2}{\Delta z}\frac{\partial \psi_1}{\partial z} \tag{13}$$

and substituting back into Equation 9 results in

$$\frac{\partial \psi_1}{\partial z} = -\frac{\Delta z}{2}\left[\frac{\partial^2 \psi_1}{\partial r^2} + 2ik_0\frac{\partial \psi_1}{\partial r} + k_0^2(n_1^2 - 1)\psi_1 - \frac{2}{(\Delta z)^2}\left(\psi_1 - \psi_{l-1}^m\right)\right]. \tag{14}$$

A similar equation can be constructed for medium 2. Equating the two, setting $\psi_1 = \psi_2 = \psi$ and satisfying the second boundary condition, Equation 11, one obtains,

$$\frac{\partial^2\psi}{\partial r^2} + 2ik_0\frac{\partial\psi}{\partial r} + k_0^2\frac{\rho_2}{\rho_1+\rho_2}\left(n_1^2 + \frac{\rho_1}{\rho_2}n_2^2\right)\psi - k_0^2\psi + \frac{2}{(\Delta z)^2}\frac{\rho_2}{\rho_1+\rho_2}\left(\psi_{l-1}^m - \frac{\rho_1+\rho_2}{\rho_2}\psi_l^m + \frac{\rho_1}{\rho_2}\psi_{l+1}^m\right) = 0 \tag{15}$$

As the equations become more complex, it is helpful to make use of additional variable substitutions. Consider:

$$\Gamma_{zz}\psi = \frac{2}{(\Delta z)^2}\frac{\rho_2}{\rho_1+\rho_2}\left(\psi_{l-1}^m - \frac{\rho_1+\rho_2}{\rho_2}\psi_l^m + \frac{\rho_1}{\rho_2}\psi_{l+1}^m\right) \tag{16}$$

$$\eta = \frac{\rho_2}{\rho_1+\rho_2}\left(n_1^2 + \frac{\rho_1}{\rho_2}n_2^2\right) - 1 \tag{17}$$

$$G = k_0^2\eta + \Gamma_{zz} \tag{18}$$

This allows one to compactly express Equation 15 as

$$\frac{\partial^2\psi}{\partial r^2} + 2ik_0\frac{\partial\psi}{\partial r} + G\psi = 0. \tag{19}$$

Setting $G = k_0^2(Q^2-1)$ and using the generalized operators described in Section 6.2.2 of Computational Ocean Acoustics, one derives

$$\begin{aligned}\frac{\partial\psi}{\partial r} &= ik_0(Q-1)\psi \\ &= ik_0\left(\sqrt{1+q}-1\right)\psi,\end{aligned} \tag{20}$$

which is the generalized parabolic wave equation valid on horizontal surfaces.

Continuing along the derivation supplied in Section 6.6.2 of Computational Ocean Acoustics, the differential equation shown above can be solved using the Crank-Nicholson finite difference scheme, as outlined by Lee and McDaniel in 1988. Assuming

$$\frac{\psi^{m+1}-\psi^m}{\Delta r} = ik_0\left(\sqrt{1+q}-1\right)\frac{\psi^{m+1}+\psi^m}{2}, \tag{21}$$

one can rearrange terms to obtain an iterative equation

$$\left[1 - \frac{ik_0\Delta r}{2}\left(\sqrt{1+q}-1\right)\right]\psi^{m+1} = \left[1 - \frac{ik_0\Delta r}{2}\left(\sqrt{1+q}-1\right)\right]\psi^m, \tag{22}$$

and apply a rational function approximation of the square root operator $\sqrt{1+q}$,

$$\sqrt{1+q} \simeq \frac{a_0 + a_1 q}{b_0 + b_1 q}, \tag{23}$$

which yields

$$\left[1 - \frac{ik_0\Delta r}{2}\left(\frac{a_0 + a_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)}{b_0 + b_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)} - 1\right)\right]\psi^{m+1} = \left[1 - \frac{ik_0\Delta r}{2}\left(\frac{a_0 + a_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)}{b_0 + b_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)} - 1\right)\right]\psi^m. \tag{24}$$

Next, assume that $b_0 + b_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)$ is constant across Δr, to obtain

$$\begin{aligned}&\left[b_0 + b_1\eta - \frac{ik_0\Delta r}{2}\left[(a_0-b_0) + (a_1-b_1)\eta\right]\right]\psi^{m+1} + \frac{1}{k_0^2}\left[b_1 - \frac{ik_0\Delta r}{2}(a_1-b_1)\right]\Gamma_{zz}\psi^{m+1} = \\ &\quad\left[b_0 + b_1\eta - \frac{ik_0\Delta r}{2}\left[(a_0-b_0) + (a_1-b_1)\eta\right]\right]\psi^m + \frac{1}{k_0^2}\left[b_1 - \frac{ik_0\Delta r}{2}(a_1-b_1)\right]\Gamma_{zz}\psi^m.\end{aligned} \tag{25}$$

One uses the following shorthand notation to further simplify the expression:

$$w_1 = b_0 + \frac{ik_0\Delta r}{2}(a_0 - b_0) \tag{26}$$

$$w_1^* = b_0 - \frac{ik_0\Delta r}{2}(a_0 - b_0) \tag{27}$$

$$w_2 = b_1 + \frac{ik_0\Delta r}{2}(a_1 - b_1) \tag{28}$$

$$w_2^* = b_1 - \frac{ik_0\Delta r}{2}(a_1 - b_1), \tag{29}$$

finally yielding

$$\left(\frac{w_1^*}{w_2^*} + \eta\right)\psi_l^{m+1} + \frac{1}{k_0^2}\left[\frac{2}{(\Delta z)^2}\frac{\rho_2}{\rho_1+\rho_2}\right] \times \left(\psi_{l-1}^{m+1} - \frac{\rho_1+\rho_2}{\rho_2}\psi_l^{m+1} + \frac{\rho_1}{\rho_2}\psi_{l+1}^{m+1}\right) = \left(\frac{w_1 + w_2\eta}{w_2^*}\right)\psi_l^m + \frac{1}{k_0^2}\left(\frac{w_2}{w_2^*}\right)\left[\frac{2}{(\Delta z)^2}\frac{\rho_2}{\rho_1+\rho_2}\right] \times \left(\psi_{l-1}^{m} - \frac{\rho_1+\rho_2}{\rho_2}\psi_l^{m} + \frac{\rho_1}{\rho_2}\psi_{l+1}^{m}\right). \tag{30}$$

Rearranging terms, one can write this in vector form as

$$[1, u, v]\begin{bmatrix} \psi_{l-1}^{m+1} \\ \psi_l^{m+1} \\ \psi_{l+1}^{m+1} \end{bmatrix} = \frac{w_2}{w_2^*}[1, \hat{u}, v]\begin{bmatrix} \psi_{l-1}^{m} \\ \psi_l^{m} \\ \psi_{l+1}^{m} \end{bmatrix}, \tag{31}$$

with

$$u = \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] + \frac{k_0^2(\Delta z)^2}{2}\left[\left(n_1^2 - 1\right) + \frac{\rho_1}{\rho_2}\left(n_2^2 - 1\right)\right] \tag{32}$$

$$\hat{u} = \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1}{w_2}\right) - 1\right] + \frac{k_0^2(\Delta z)^2}{2}\left[\left(n_1^2 - 1\right) + \frac{\rho_1}{\rho_2}\left(n_2^2 - 1\right)\right] \tag{33}$$

$$v = \frac{\rho_1}{\rho_2} \tag{34}$$

Equations 31 through 34 represent the starting point for the derived distributions. Marched through range and depth, they express the pressure field envelope ψ as a function of the refractive index, n. This represents the last of the background material from Computational Ocean Acoustics. We now proceed with derived distributions.

Derived Distributions

Before proceeding further, it is useful to review the concept of derived distributions. Given a random variable x, with probability density function (PDF) $p_x(x)$, and a function $y = g(x)$, one wishes to find the probability density function $p_y(y)$ of y.

From Section 5-2 of Papoulis, one must first express x as a function of y. If the function is not one-to-one, one must account for all possible roots of x,

$$y = g(x_1) = \cdots = g(x_n) \tag{35}$$

and then the PDF of y can be calculated from

$$p_y(y) = \frac{p_x(x_1)}{\left|\frac{\partial}{\partial x}g(x)\big|_{x=x_1}\right|} + \cdots + \frac{p_x(x_n)}{\left|\frac{\partial}{\partial x}g(x)\big|_{x=x_n}\right|} \tag{36}$$

A few examples will illustrate the concept.

- $y = ax + b$: Here, a and b are scalar deterministic variables. Assuming $y = g(x)$, x is uniquely determined by y, and vice-versa.

$$x_1 = \frac{y-b}{a} \qquad \frac{\partial y}{\partial x} = a \tag{37}$$

Application of Equation 36 yields:

$$p_y(y) = \frac{1}{|a|} p_x \left(\frac{y-b}{a} \right). \tag{38}$$

- $y = \frac{a^2}{x^2}$: a, b are scalar deterministic variables. x can take two values given a particular y,

$$x_1 = -\frac{a}{\sqrt{y}} \qquad x_2 = +\frac{a}{\sqrt{y}} \tag{39}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x=x_1} = \frac{2}{a} y^{\frac{3}{2}} \qquad \left.\frac{\partial y}{\partial x}\right|_{x=x_2} = -\frac{2}{a} y^{\frac{3}{2}} \tag{40}$$

Substitution of Equation 36 results in:

$$p_y(y) = \frac{a}{\left|2y^{\frac{3}{2}}\right|} p_x \left(-\frac{a}{\sqrt{y}} \right) + \frac{a}{\left|2y^{\frac{3}{2}}\right|} p_x \left(\frac{a}{\sqrt{y}} \right) \tag{41}$$

For a case where there are two random variables (x, y) and two functions of these two random variables, one can derive a new joint distribution based on their joint PDF, $p_{xy}(x, y)$. From Papoulis, given

$$z = g(x, y) \qquad w = h(x, y), \tag{42}$$

one solves for x and y in terms of z and w. If the relationship between the two pairs of variables is not one-to-one, all possible roots (x_n, y_n) should be considered. The joint distribution $p_{zw}(z, w)$ can be expressed as

$$p_{zw}(z, w) = \frac{p_{xy}(x_1, y_1)}{|J(x_1, y_1)|} + \cdots + \frac{p_{xy}(x_n, y_n)}{|J(x_n, y_n)|}, \tag{43}$$

where $J(x, y)$ is the Jacobian operator,

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}. \tag{44}$$

An example will help illustrate the procedure.

- Given $p_{xy}(x, y)$, and the two functions

$$z = ax + by + c \qquad w = x \tag{45}$$

one wishes to solve for $p_{zw}(z, w)$. The equations are one-to-one, with resulting inverses,

$$x = w \qquad y = \frac{1}{b}(z - aw - c). \tag{46}$$

The Jacobian can be easily evaluated as $J(x, y) = -b$; these result in a derived distribution of

$$p_{zw}(z, w) = \frac{1}{b} p_{xy} \left(w, \frac{1}{b} [z - aw - c] \right) \tag{47}$$

- Given $p_{xy}(x,y)$, and the two functions

$$z = ay + bx^{-1}y + cx^{-1} \qquad w = x \tag{48}$$

one notes the equations are one-to-one, resulting in a single set of inverse equations,

$$x = w \qquad y = \frac{z - cw^{-1}}{a + bw^{-1}}. \tag{49}$$

One can then solve for the Jacobian,

$$\begin{array}{ll} \dfrac{\partial z}{\partial x} = -bx^{-2}y - cx^{-2} & \dfrac{\partial z}{\partial y} = a + bx^{-1} \\ \dfrac{\partial w}{\partial x} = 1 & \dfrac{\partial w}{\partial y} = 0 \end{array} \quad \text{with} \qquad J(x,y) = \left| \begin{array}{cc} -bx^{-2}y - cx^{-2} & a + bx^{-1} \\ 1 & 0 \end{array} \right| \tag{50}$$

$$J(x,y) = \left| a + bx^{-1} \right|. \tag{51}$$

Thus the derived distribution becomes

$$p_{zw}(z,w) = \frac{p_{xy}\left(w, \frac{z - cw^{-1}}{a + bw^{-1}}\right)}{|a + bw^{-1}|}. \tag{52}$$

- Given $p_{xy}(x,y)$, and the two independent functions,

$$z = a^2x^{-2} \qquad w = a^2y^{-2}, \tag{53}$$

solve for $p_{zw}(z,w)$. These equations are not one-to-one, with four different roots as

$$x = \pm\frac{a}{\sqrt{z}} \qquad y = \pm\frac{a}{\sqrt{z}}, \tag{54}$$

and the Jacobian evaluated to be $J(x,y) = 4a^4x^{-3}y^{-3}$, resulting in

$$\begin{aligned} p_{zw}(z,w) = \frac{1}{4a^4} & \left| \left(\frac{a}{\sqrt{z}}\right)^3 \left(\frac{a}{\sqrt{z}}\right)^3 \right| \\ \times \quad & \left[p_{xy}\left(-\frac{a}{\sqrt{z}}, -\frac{a}{\sqrt{w}}\right) + p_{xy}\left(-\frac{a}{\sqrt{z}}, \frac{a}{\sqrt{w}}\right) + p_{xy}\left(\frac{a}{\sqrt{z}}, -\frac{a}{\sqrt{w}}\right) + p_{xy}\left(\frac{a}{\sqrt{z}}, \frac{a}{\sqrt{w}}\right) \right]. \end{aligned} \tag{55}$$

With this background information it is possible to derive the PDF of the output pressure field, $p_p(p)$, given the PDF of the input sound velocity profile $p_c(c)$, in the trivial isovelocity case.

Trivial Isovelocity Case

In this simplified case, the environment is taken to be range-independent: ρ_1, ρ_2 are constants, as is the sound speed, $c = c_1 = c_2$ and $n = n_1 = n_2$. The top and bottom pressure envelopes, ψ^m_{l-1} and ψ^m_{l+1} are known and constant for all r (and m). One assumes the initial field envelope, ψ^0_l and its distribution, $p_{\psi_0}(\psi^0_l)$ is known, and is independent of the sound speed distribution, $p_c(c)$.

Starting with the input sound velocity PDF $p_c(c)$ one wishes to find the PDF for the output pressure p at a point in the middle of the water column.

The first step is to express the PDF of the refractive index, $p_n(n)$, as a function of $p_c(c)$. The refractive index is related to the sound speed by:

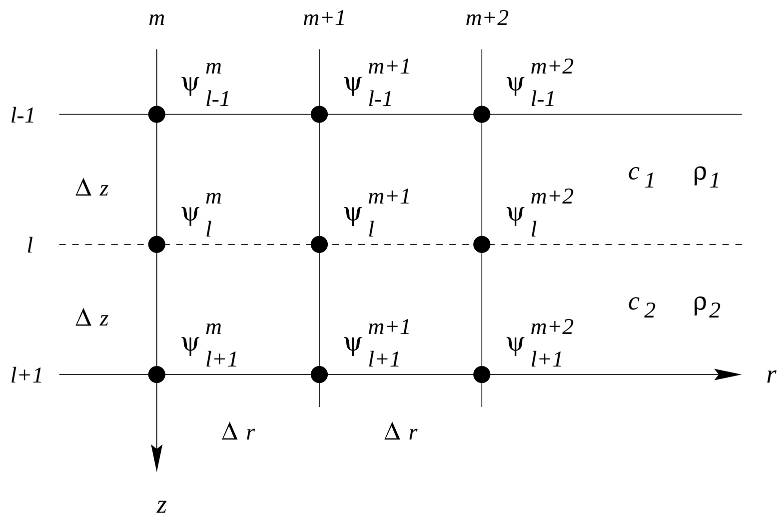

Figure 2: Plot of finite difference cell for trivial isovelocity case. Here, $c = c_1 = c_2$.

$$n^2 = \frac{c_0^2}{c^2}, \tag{56}$$

with c_0 a reference sound speed. Using Equation 41, with n^2 in place of y, c_0 as a, and c as x,

$$p_{n^2}(n^2) = \frac{c_0}{2\,(n^2)^{\frac{3}{2}}}\left[p_c\left(-\frac{c_0}{\sqrt{n^2}}\right) + p_c\left(+\frac{c_0}{\sqrt{n^2}}\right)\right]. \tag{57}$$

Recalling Equations 31 through 34, we will assume $n_1 = n_2 = n$, and rewrite Equation 32 as

$$\begin{aligned} u &= \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] + \frac{k_0^2(\Delta z)^2}{2}\left[\left(n_1^2-1\right) + \frac{\rho_1}{\rho_2}\left(n_2^2-1\right)\right] \\ &= \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] + \frac{k_0^2(\Delta z)^2}{2}\left[\left(1+\frac{\rho_1}{\rho_2}\right)\left(n^2-1\right)\right] \\ &= \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] - \frac{k_0^2(\Delta z)^2}{2}\left(1+\frac{\rho_1}{\rho_2}\right) + \frac{k_0^2(\Delta z)^2}{2}n^2 \end{aligned} \tag{58}$$

Substituting

$$\alpha = \frac{k_0^2(\Delta z)^2}{2} \tag{59}$$

$$\beta = \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] - \frac{k_0^2(\Delta z)^2}{2}\left(1+\frac{\rho_1}{\rho_2}\right) \tag{60}$$

yields

$$u = \alpha n^2 + \beta \tag{61}$$

Application of Equation 38 gives

$$\begin{aligned} p_u(u) &= \frac{1}{\alpha}p_{n^2}\left(\frac{u-\beta}{\alpha}\right) \\ &= \frac{2}{k_0^2(\Delta z)^2\left(1+\frac{\rho_1}{\rho_2}\right)} \times p_{n^2}\left(\frac{u - \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right)-1\right] + \frac{k_0^2(\Delta z)^2}{2}\left(1+\frac{\rho_1}{\rho_2}\right)}{\frac{k_0^2(\Delta z)^2}{2}\left(1+\frac{\rho_1}{\rho_2}\right)}\right) \end{aligned} \tag{62}$$

A similar PDF can be constructed for $\hat{u}$ by substituting w_1 for w_1^* and w_2 for w_2^*. However, this results in two correlated variables in Equation 31, increasing the complexity uneccessarily. Instead, it is better to rewrite $\hat{u}$ in terms of u. Assuming:

$$\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} \right) - 1 \right] - \frac{k_0^2 (\Delta z)^2}{2} \left(1 + \frac{\rho_1}{\rho_2} \right) \quad (63)$$

$$\hat{\beta} = \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1}{w_2} \right) - 1 \right] - \frac{k_0^2 (\Delta z)^2}{2} \left(1 + \frac{\rho_1}{\rho_2} \right) \quad (64)$$

with

$$u = \alpha n^2 + \beta \quad (65)$$

$$\hat{u} = \alpha n^2 + \hat{\beta} \quad (66)$$

$$\Delta\beta = \hat{\beta} - \beta \quad (67)$$

so

$$\hat{u} = u + \Delta\beta \quad (68)$$

Rewriting Equation 31 to take advantage of $\Delta\beta$,

$$[1, u, v] \begin{bmatrix} \psi_{l-1}^{m+1} \\ \psi_l^{m+1} \\ \psi_{l+1}^{m+1} \end{bmatrix} = \frac{w_2}{w_2^*} [1, u + \Delta\beta, v] \begin{bmatrix} \psi_{l-1}^{m} \\ \psi_l^{m} \\ \psi_{l+1}^{m} \end{bmatrix}, \quad (69)$$

and isolating the unknowns ψ_l^{m+1} and u,

$$\begin{aligned} \psi_l^{m+1} &= \frac{w_2}{w_2^*} \psi_l^m + \frac{1}{u} \left\{ \frac{w_2}{w_2^*} \left[\psi_{l-1}^m + \Delta\beta \psi_l^m + v \psi_{l+1}^m \right] - \psi_{l-1}^{m+1} - v \psi_{l+1}^{m+1} \right\} \\ &= \frac{w_2}{w_2^*} \psi_l^m + \frac{1}{u} \Delta\beta \psi_l^m + \frac{1}{u} \left\{ \frac{w_2}{w_2^*} \left[\psi_{l-1}^m + v \psi_{l+1}^m \right] - \left[\psi_{l-1}^{m+1} + v \psi_{l+1}^{m+1} \right] \right\} \quad (70) \end{aligned}$$

Substituting the variables

$$\mathcal{A} = \frac{w_2}{w_2^*} \quad (71)$$

$$\mathcal{B} = \Delta\beta \quad (72)$$

$$\mathcal{C} = \frac{w_2}{w_2^*} \left[\psi_{l-1}^m + v \psi_{l+1}^m \right] - \left[\psi_{l-1}^{m+1} + v \psi_{l+1}^{m+1} \right] \quad (73)$$

into Equation 70 gives

$$\psi_l^{m+1} = \mathcal{A} \psi_l^m + \frac{1}{u} \mathcal{B} \psi_l^m + \frac{1}{u} \mathcal{C}. \quad (74)$$

Application of Equation 52 yields the derived distribution

$$p_{u, \psi_l^{m+1}}(u, \psi_l^{m+1}) = \frac{p_{u, \psi_l^m}(u, \psi_l^m) \left(u, \frac{\psi_l^{m+1} - \mathcal{C} u^{-1}}{\mathcal{A} + \mathcal{B} u^{-1}} \right)}{|\mathcal{A} + \mathcal{B} u^{-1}|}. \quad (75)$$

Once one reaches the desired range r (iteration m), one can calculate the marginal distribution of ψ_l^{m+1},

$$p_{\psi_l^{m+1}}(\psi_l^{m+1}) = \int_{-\infty}^{+\infty} p_{u, \psi_l^{m+1}}(u, \psi_l^{m+1}) du. \quad (76)$$

The final step expresses the complex pressure field, p as a function of the envelope, ψ. Using the far-field Hankel approximation,

$$p(r,z) = \frac{\psi(r,z)}{\sqrt{r}} e^{i(k_0 r - \frac{\pi}{4})} \tag{77}$$

with Equation 38 gives

$$p_p(p) = \sqrt{r} \times p_\psi \left(\sqrt{r} e^{-i(k_0 r - \frac{\pi}{4})} p \right). \tag{78}$$

Two speed case

The logical extension to the isovelocity case is one where the sound speeds c_1 and c_2 differ. To model the effect on the pressure field envelope ψ_l^m, one would need to know the joint PDF of the two sound speed layers, $p_{c_1 c_2}(c_1, c_2)$.

The first step would be to calculate the joint PDF of $p_{n_1 n_2}(n_1^2, n_2^2)$ in terms of $p_{c_1 c_2}(c_1, c_2)$. Recall the relationship between n^2 and c is $n^2 = (c_0/c)^2$, where c_0 is a known reference sound speed. Using Equation 55 with $a = c_0, x = c_1, y = c_2, z = n_1^2$, and $w = n_2^2$, one calculates:

$$\begin{aligned} p_{n_1 n_2}(n_1^2, n_2^2) = \frac{c_0^2}{4} \left[\frac{1}{(n_1^2 n_2^2)^{\frac{3}{2}}} \right] \\ \times \quad \left[p_{c_1 c_2} \left(-\frac{c_0}{\sqrt{n_1^2}}, -\frac{c_0}{\sqrt{n_2^2}} \right) + p_{c_1 c_2} \left(-\frac{c_0}{\sqrt{n_1^2}}, \frac{c_0}{\sqrt{n_2^2}} \right) \right. \\ \left. + p_{c_1 c_2} \left(\frac{c_0}{\sqrt{n_1^2}}, -\frac{c_0}{\sqrt{n_2^2}} \right) + p_{c_1 c_2} \left(\frac{c_0}{\sqrt{n_1^2}}, \frac{c_0}{\sqrt{n_2^2}} \right) \right]. \end{aligned} \tag{79}$$

Integrating this with the isovelocity derivation, the next step is to find the PDF $p_u(u)$ in terms of the joint PDF $p_{n_1 n_2}(n_1, n_2)$. Starting with Equation 32, and isolating u, n_1 and n_2,

$$\begin{aligned} u &= \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} \left[(n_1^2 - 1) + \frac{\rho_1}{\rho_2} \left(n_2^2 - 1 \right) \right] \\ &= \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} - 1 \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} n_1^2 + \frac{k_0^2 (\Delta z)^2}{2} \frac{\rho_1}{\rho_2} n_2^2 \end{aligned} \tag{80}$$

Setting

$$\alpha_1 = \frac{k_0^2 (\Delta z)^2}{2} \tag{81}$$

$$\alpha_2 = \frac{k_0^2 (\Delta z)^2}{2} \frac{\rho_1}{\rho_2} \tag{82}$$

$$\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} - 1 \right) - 1 \right] \tag{83}$$

one can rewrite Equation 80 as

$$u = \alpha_1 n_1^2 + \alpha_2 n_2^2 + \beta. \tag{84}$$

Application of Equation 47 with $a = \alpha_1$, $b = \alpha_2$, $c = \beta$, $x = n_1^2$, $y = n_2^2$, and $z = u$, one finds

$$p_{u n_1}(u, n_1^2) = \frac{1}{\alpha_2} p_{n_1 n_2}(n_1^2, n_2^2) \left(n_1^2, \frac{1}{\alpha_2} \left[u - \alpha_1 n_1^2 - \beta \right] \right). \tag{85}$$

The marginal distribution $p_u(u)$ can be calculated by integrating with respect to n_1^2,

$$p_u(u) = \frac{1}{\alpha_2} \int_{-\infty}^{+\infty} p_{un_1}(u, q) \left(q, \frac{1}{\alpha_2}[u - \alpha_1 q - \beta] \right) dq. \tag{86}$$

The derivation would follow the isovelocity case, continuing with Equation 67, except β and $\hat{\beta}$ would be defined as

$$\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} - 1 \right) - 1 \right] \tag{87}$$

$$\hat{\beta} = \frac{\rho_1 + \rho_2}{\rho_2} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1}{w_2} - 1 \right) - 1 \right]. \tag{88}$$

N mesh points

Increasing the number of vertical mesh points allows one to improve the resolution and accuracy of both the simulated pressure field and the derived statistics.

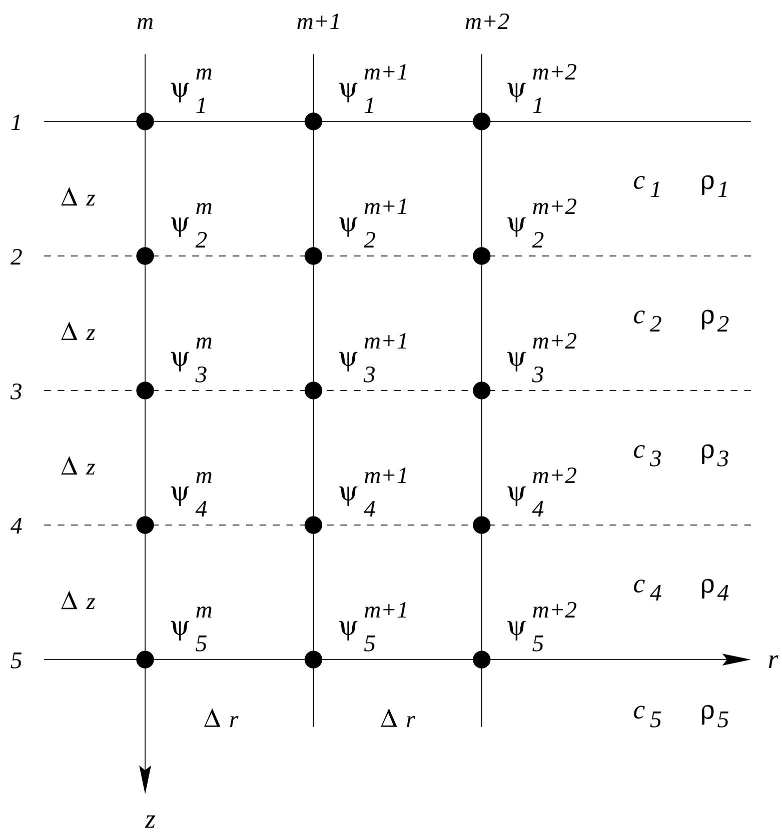

Figure 3: Plot of finite difference cells for 5 mesh points.

Figure 3 illustrates the N vertical mesh point case with five points. The first row of points, ψ_1^m, is assumed deterministic and known. This represents the air-water boundary. Additionally, the environment is assumed range-independent; the statistics of c_l do not vary with range. Density, ρl is assumed known and deterministic. The environment can be any number of mesh points; it is limited to five here for brevity.

With the parabolic equation method, one steps through range, solving for the pressure field envelope at each mesh point. Solving for the PDF is similar; one recursively derives the joint PDF for each range and depth point, marching in range. Every attempt was made to keep the resulting expressions as simple as possible. Unfortunately, the recursive nature of the solution does not lend itself to a closed form expression. Instead, using these equations to numerically evaluate the PDF is recommended.

We start assuming the joint PDF of the sound speed profile, $p_{\mathbf{c}}(\mathbf{c})$ is known. One uses the relationship

$$\nu_l = n_l^2 = \frac{c_0^2}{c_l^2} \tag{89}$$

to calculate the joint PDF of $p_\nu(\nu)$. Using the formula for derived distribution,

$$p_\nu(\nu) = \frac{p_{\mathbf{c}}(\mathbf{c})}{|J(\mathbf{c})|}, \tag{90}$$

one must calculate the roots of Equation 89 and the Jacobian $\mathbf{c}$. The roots can be calculated for each element of ν as:

$$c_l = \pm \frac{c_0}{\sqrt{\nu_l}} \tag{91}$$

The $\pm$ term is problematic, as it shows the relationship between the two equations is not one-to-one. Rather, one must include both terms when expressing the distribution. Taking the Jacobian first,

$$\begin{aligned} J(\mathbf{c}) &= \begin{vmatrix} \frac{\partial \nu_1}{\partial c_1} & \cdots & \frac{\partial \nu_1}{\partial c_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial \nu_L}{\partial c_1} & \cdots & \frac{\partial \nu_L}{\partial c_L} \end{vmatrix} = \begin{vmatrix} -2c_0 c_1^{-3} & & 0 \\ & \ddots & \\ 0 & & -2c_0 c_L^{-3} \end{vmatrix} \\ &= (-2c_0)^L \prod_{l=1}^{L} c_l^{-3} \end{aligned} \tag{92}$$

Substitution of Equation 91, and taking the magnitude of the result gives

$$|J(\mathbf{c})| = 2^L c_0^{-2L} \prod_{l=1}^{L} \nu_l^{3/2}. \tag{93}$$

Evaluation of Equation 90 requires summation over 2^L terms,

$$p_\nu(\nu) = \sum_{m=1}^{2^L} \frac{p_c(\mathbf{\Upsilon}_m)}{2^L c_0^{-2L} \prod_{l=1}^{L} \nu_l^{3/2}} \tag{94}$$

$$\text{where:} \qquad \mathbf{\Upsilon}_m = c_0 \times \begin{bmatrix} \pm\nu_1^{-1/2} \\ \vdots \\ \pm\nu_L^{-1/2} \end{bmatrix},$$

and the sign of each element in $\mathbf{\Upsilon}_m$ is determined by the binary value of m, with each binary digit assigned to its corresponding element of $\mathbf{\Upsilon}_m$.

The second step is to solve for the joint PDF of $p_{\mathbf{u}}(\mathbf{u})$ given $p_\nu(\nu)$. Recall

$$u_l = \frac{\rho_{l-1} + \rho_l}{\rho_l} \left[\frac{k_0^2 (\Delta z)^2}{2} \left(\frac{w_1^*}{w_2^*} \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} \left[(\nu_{l-1} - 1) + \frac{\rho_{l-1}}{\rho_l} (\nu_l - 1) \right] \tag{95}$$

can be expressed as

$$u_l = \alpha \nu_{l-1} + \beta_l \nu_l + \gamma_l \tag{96}$$

with the appropriate substitutions. Using the derived distribution approach outlined above, the Jacobian evaluates as

$$J(\nu) = \prod_{l=1}^{L} \beta_l. \tag{97}$$

Inversion of Equation 95 to solve for the roots of ν_l gives

$$\nu_l = \frac{1}{\beta_l}\left[u_l - \alpha\nu_{l-1} - \gamma_l\right]. \tag{98}$$

This equation requires one solves for ν_l recursively. One can assume ν_0 is a known deterministic value. With this information, one can express the PDF of $p_{\mathbf{u}}(\mathbf{u})$ as

$$p_{\mathbf{u}}(\mathbf{u}) = \frac{p_\nu(\nu)}{\left|\prod_{l=1}^{L}\beta_l\right|}, \tag{99}$$

with the elements of ν determined by Equation 98.

With the PDF of $\mathbf{u}$, one can calculate the iterative PDF for the pressure field envelope, $\mathbf{\Psi}$. One starts assuming the joint PDF for the previous range step is known. Letting $\mathbf{\Psi}^m = \left[\Psi_2^m \Psi_3^m \cdots \Psi_L^m\right]^T$ and $\mathbf{u} = \left[u_1 u_2 \cdots u_L\right]^T$, one can assume the joint PDF of these two random vectors is known: $p_{\mathbf{\Psi}u}(\mathbf{\Psi}^m, \mathbf{u})$. At the source, the PDF of $\mathbf{u}$ and $\mathbf{\Psi}^0$ are assumed to be independent; this results in their initial joint PDF to be

$$p_{\mathbf{\Psi}u}(\mathbf{\Psi}^0, \mathbf{u}) = p_{\mathbf{\Psi}}(\mathbf{\Psi}^0) p_u(\mathbf{u}). \tag{100}$$

Calculation of the pressure field envelope vector, $\mathbf{\Psi}^{m+1}$ is performed by solving the system of tridiagonal matrices,

$$\begin{bmatrix} u_1 & v_1 & & & & & \\ 1 & u_2 & v_2 & & & & \\ & 1 & u_3 & v_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & u_{L-2} & v_{L-2} & \\ & & & & 1 & u_{L-1} & v_{L-1} \\ & & & & & 1 & u_L \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{L-2} \\ \psi_{L-1} \\ \psi_L \end{bmatrix}^{m+1}$$
$$= \xi \begin{bmatrix} \hat{u}_1 & v_1 & & & & & \\ 1 & \hat{u}_2 & v_2 & & & & \\ & 1 & \hat{u}_3 & v_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & \hat{u}_{L-2} & v_{L-2} & \\ & & & & 1 & \hat{u}_{L-1} & v_{L-1} \\ & & & & & 1 & \hat{u}_L \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{L-2} \\ \psi_{L-1} \\ \psi_L \end{bmatrix}^{m}, \tag{101}$$

where:

$$\xi = \frac{w_2}{w_2^*} \tag{102}$$
$$\hat{u}_l = u_l - \Delta u_l \tag{103}$$
$$\Delta u_l = \frac{\rho_{l-1} + \rho_l}{\rho_l}\frac{k_0^2(\Delta z)^2}{2}\left[\frac{w_1^*}{w_2^*} - \frac{w_1}{w_2}\right] \tag{104}$$
$$\text{and} \quad v_l = \frac{\rho_{l-1}}{\rho_l}. \tag{105}$$

Evaluation of the tridiagonal system is straightforward. As the top layer ψ_1^{m+1} is assumed known, subsequent layers, starting with ψ_2^{m+1} can be calculated,

$$\begin{aligned}
\psi_2^{m+1} &= -\frac{u_1}{v_1}\psi_1^{m+1} + \xi\frac{\hat{u}_1}{v_1}\psi_1^m + \xi\psi_2^m && (106)\\
\psi_3^{m+1} &= -\frac{1}{v_2}\psi_1^{m+1} - \frac{u_2}{v_2}\psi_2^{m+1} + \frac{\xi}{v_2}\psi_1^m + \xi\frac{\hat{u}_2}{v_2}\psi_2^m + \xi\psi_3^m && (107)\\
\psi_4^{m+1} &= -\frac{1}{v_3}\psi_2^{m+1} - \frac{u_3}{v_3}\psi_3^{m+1} + \frac{\xi}{v_3}\psi_2^m + \xi\frac{\hat{u}_3}{v_3}\psi_3^m + \xi\psi_4^m && (108)\\
&\vdots \\
\psi_l^{m+1} &= -\frac{1}{v_{l-1}}\psi_{l-2}^{m+1} - \frac{u_{l-1}}{v_{l-1}}\psi_{l-1}^{m+1} + \frac{\xi}{v_{l-1}}\psi_{l-2}^m + \xi\frac{\hat{u}_{l-1}}{v_{l-1}}\psi_{l-1}^m + \xi\psi_l^m. && (109)
\end{aligned}$$

With this information a derived distribution can be calculated. The set of equations can be expressed in using partitioned vectors,

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) \tag{110}$$

with $\mathbf{\Psi}^m = [\psi_2^m \psi_3^m \cdots \psi_L^m]^T$,

$$\mathbf{y} = \begin{bmatrix} \mathbf{\Psi}^{m+1} \\ \mathbf{u} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \mathbf{\Psi}^{m} \\ \mathbf{u} \end{bmatrix} \tag{111}$$

and $\mathbf{g}(\mathbf{x})$ utilizes Equations 106 through 109. The derived distribution takes the form:

$$p_y(\mathbf{y}) = \frac{p_x(\mathbf{x})}{|J(\mathbf{x})|} \tag{112}$$

where $J(\mathbf{x})$ is the Jacobian operator, evaluated as

$$J(\mathbf{x}) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}. \tag{113}$$

To evaluate the derived distribution, one must solve both Equation 113 and $\mathbf{x} = \mathbf{g}^{-1}(\mathbf{y})$ for Equation 112. In the example 5 mesh point case, the Jacobian can be expressed as

$$J\left(\left[\begin{array}{c}\boldsymbol{\Psi}^m\\ \mathbf{u}\end{array}\right]\right)=\left|\begin{array}{ccccccccc}
\frac{\partial\psi_2^{m+1}}{\partial\psi_2^m} & \frac{\partial\psi_2^{m+1}}{\partial\psi_3^m} & \frac{\partial\psi_2^{m+1}}{\partial\psi_4^m} & \frac{\partial\psi_2^{m+1}}{\partial\psi_5^m} & \frac{\partial\psi_2^{m+1}}{\partial u_1} & \frac{\partial\psi_2^{m+1}}{\partial u_2} & \frac{\partial\psi_2^{m+1}}{\partial u_3} & \frac{\partial\psi_2^{m+1}}{\partial u_4} & \frac{\partial\psi_2^{m+1}}{\partial u_5}\\
\frac{\partial\psi_3^{m+1}}{\partial\psi_2^m} & \frac{\partial\psi_3^{m+1}}{\partial\psi_3^m} & \frac{\partial\psi_3^{m+1}}{\partial\psi_4^m} & \frac{\partial\psi_3^{m+1}}{\partial\psi_5^m} & \frac{\partial\psi_3^{m+1}}{\partial u_1} & \frac{\partial\psi_3^{m+1}}{\partial u_2} & \frac{\partial\psi_3^{m+1}}{\partial u_3} & \frac{\partial\psi_3^{m+1}}{\partial u_4} & \frac{\partial\psi_3^{m+1}}{\partial u_5}\\
\frac{\partial\psi_4^{m+1}}{\partial\psi_2^m} & \frac{\partial\psi_4^{m+1}}{\partial\psi_3^m} & \frac{\partial\psi_4^{m+1}}{\partial\psi_4^m} & \frac{\partial\psi_4^{m+1}}{\partial\psi_5^m} & \frac{\partial\psi_4^{m+1}}{\partial u_1} & \frac{\partial\psi_4^{m+1}}{\partial u_2} & \frac{\partial\psi_4^{m+1}}{\partial u_3} & \frac{\partial\psi_4^{m+1}}{\partial u_4} & \frac{\partial\psi_4^{m+1}}{\partial u_5}\\
\frac{\partial\psi_5^{m+1}}{\partial\psi_2^m} & \frac{\partial\psi_5^{m+1}}{\partial\psi_3^m} & \frac{\partial\psi_5^{m+1}}{\partial\psi_4^m} & \frac{\partial\psi_5^{m+1}}{\partial\psi_5^m} & \frac{\partial\psi_5^{m+1}}{\partial u_1} & \frac{\partial\psi_5^{m+1}}{\partial u_2} & \frac{\partial\psi_5^{m+1}}{\partial u_3} & \frac{\partial\psi_5^{m+1}}{\partial u_4} & \frac{\partial\psi_5^{m+1}}{\partial u_5}\\
\frac{\partial u_1}{\partial\psi_2^m} & \frac{\partial u_1}{\partial\psi_3^m} & \frac{\partial u_1}{\partial\psi_4^m} & \frac{\partial u_1}{\partial\psi_5^m} & \frac{\partial u_1}{\partial u_1} & \frac{\partial u_1}{\partial u_2} & \frac{\partial u_1}{\partial u_3} & \frac{\partial u_1}{\partial u_4} & \frac{\partial u_1}{\partial u_5}\\
\frac{\partial u_2}{\partial\psi_2^m} & \frac{\partial u_2}{\partial\psi_3^m} & \frac{\partial u_2}{\partial\psi_4^m} & \frac{\partial u_2}{\partial\psi_5^m} & \frac{\partial u_2}{\partial u_1} & \frac{\partial u_2}{\partial u_2} & \frac{\partial u_2}{\partial u_3} & \frac{\partial u_2}{\partial u_4} & \frac{\partial u_2}{\partial u_5}\\
\frac{\partial u_3}{\partial\psi_2^m} & \frac{\partial u_3}{\partial\psi_3^m} & \frac{\partial u_3}{\partial\psi_4^m} & \frac{\partial u_3}{\partial\psi_5^m} & \frac{\partial u_3}{\partial u_1} & \frac{\partial u_3}{\partial u_2} & \frac{\partial u_3}{\partial u_3} & \frac{\partial u_3}{\partial u_4} & \frac{\partial u_3}{\partial u_5}\\
\frac{\partial u_4}{\partial\psi_2^m} & \frac{\partial u_4}{\partial\psi_3^m} & \frac{\partial u_4}{\partial\psi_4^m} & \frac{\partial u_4}{\partial\psi_5^m} & \frac{\partial u_4}{\partial u_4} & \frac{\partial u_4}{\partial u_2} & \frac{\partial u_4}{\partial u_3} & \frac{\partial u_4}{\partial u_4} & \frac{\partial u_4}{\partial u_5}\\
\frac{\partial u_5}{\partial\psi_2^m} & \frac{\partial u_5}{\partial\psi_3^m} & \frac{\partial u_5}{\partial\psi_4^m} & \frac{\partial u_5}{\partial\psi_5^m} & \frac{\partial u_5}{\partial u_1} & \frac{\partial u_5}{\partial u_2} & \frac{\partial u_5}{\partial u_3} & \frac{\partial u_5}{\partial u_4} & \frac{\partial u_5}{\partial u_5}
\end{array}\right| \tag{114}$$

which evaluates as

$$J\left(\left[\begin{array}{c}\boldsymbol{\Psi}^m\\ \mathbf{u}\end{array}\right]\right)=\left|\begin{array}{ccccccccc}
\xi & 0 & 0 & 0 & \zeta_1 & 0 & 0 & 0 & 0\\
\xi\frac{\hat{u}_2}{v_2} & \xi & 0 & 0 & 0 & \zeta_2 & 0 & 0 & 0\\
\frac{\xi}{v_3} & \xi\frac{\hat{u}_3}{v_3} & \xi & 0 & 0 & 0 & \zeta_3 & 0 & 0\\
0 & \frac{\xi}{v_4} & \xi\frac{\hat{u}_4}{v_4} & \xi & 0 & 0 & 0 & \zeta_4 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|, \tag{115}$$

$$\text{where:}\qquad \zeta_l=\frac{1}{v_l}\psi_l^{m+1}+\frac{\xi}{v_l}\psi_l^m. \tag{116}$$

One can divide the matrix up to solve for its determinant, using the identity

$$|\mathsf{X}|=\det\left[\mathsf{X}_{11}-\mathsf{X}_{12}\mathsf{X}_{22}^{-1}\mathsf{X}_{21}\right]\det\left[\mathsf{X}_{22}\right] \tag{117}$$

$$\mathsf{X}=\left[\begin{array}{cc}\mathsf{X}_{11} & \mathsf{X}_{12}\\ \mathsf{X}_{21} & \mathsf{X}_{22}\end{array}\right] \tag{118}$$

$$
\mathsf{X}_{11} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ \xi\dfrac{\hat{u}_2}{v_2} & \xi & 0 & 0 \\ \dfrac{\xi}{v_3} & \xi\dfrac{\hat{u}_3}{v_3} & \xi & 0 \\ 0 & \dfrac{\xi}{v_4} & \xi\dfrac{\hat{u}_4}{v_4} & \xi \end{bmatrix} \quad \mathsf{X}_{12} = \begin{bmatrix} \zeta_1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & 0 \end{bmatrix} \tag{119}
$$

$$
\mathsf{X}_{21} = 0 \qquad \mathsf{X}_{22} = \mathsf{I}
$$

The determinant of triangular matrix X_{11} is the product of its diagonal elements. In this example, the determinant evaluates to ξ^4. For the more general case with L mesh points, one finds

$$
J\left(\begin{bmatrix} \mathbf{\Psi}^m \\ \mathbf{u} \end{bmatrix}\right) = \xi^{L-1} \tag{120}
$$

Solving Equation 113 and $\mathbf{x} = \mathbf{g}^{-1}(\mathbf{y})$ requires one to express ψ_l^m in terms of ψ_l^{m+1}, using the tridiagonal system of Equation 101:

$$
\begin{aligned}
\psi_2^m &= \frac{u_1}{\xi v_1}\psi_1^{m+1} + \frac{1}{\xi}\psi_2^{m+1} - \frac{\hat{u}_1}{v_1}\psi_1^m && (121)\\
\psi_3^m &= \frac{1}{\xi v_2}\psi_1^{m+1} + \frac{u_2}{\xi v_2}\psi_2^{m+1} + \frac{1}{\xi}\psi_3^{m+1} - \frac{1}{v_2}\psi_1^m - \frac{\hat{u}_2}{v_2}\psi_2^m && (122)\\
\psi_4^m &= \frac{1}{\xi v_3}\psi_2^{m+1} + \frac{u_3}{\xi v_3}\psi_3^{m+1} + \frac{1}{\xi}\psi_4^{m+1} - \frac{1}{v_3}\psi_2^m - \frac{\hat{u}_3}{v_3}\psi_3^m && (123)\\
&\vdots \\
\psi_l^m &= \frac{1}{\xi v_{l-1}}\psi_{l-2}^{m+1} + \frac{u_{l-1}}{\xi v_{l-1}}\psi_{l-1}^{m+1} + \frac{1}{\xi}\psi_l^{m+1} - \frac{1}{v_{l-1}}\psi_{l-2}^m - \frac{\hat{u}_{l-1}}{v_{l-1}}\psi_{l-1}^m && (124)
\end{aligned}
$$

With this information, a recursive expression can be established for the derived joint PDF.

Given the joint PDF $p_{\mathbf{\Psi u}}(\mathbf{\Psi}^m, \mathbf{u})$, one can solve for the joint PDF $p_{\mathbf{\Psi u}}(\mathbf{\Psi}^{m+1}, \mathbf{u})$:

$$
p_{\mathbf{\Psi u}}(\mathbf{\Psi}^{m+1}, \mathbf{u}) = \frac{1}{|\xi^{L-1}|} p_{\mathbf{\Psi u}}(\mathbf{\Phi}, \mathbf{u}), \tag{125}
$$

where $\mathbf{\Phi}$ is a column vector with entries $2 \leq l \leq L$ which satisfy

$$
\phi_l = \begin{cases} \dfrac{u_1}{\xi v_1}\psi_1^{m+1} + \dfrac{1}{\xi}\psi_2^{m+1} - \dfrac{\hat{u}_1}{v_1}\psi_1^m & \text{for } l = 2 \\ \dfrac{1}{\xi v_{l-1}}\psi_{l-2}^{m+1} + \dfrac{u_{l-1}}{\xi v_{l-1}}\psi_{l-1}^{m+1} + \dfrac{1}{\xi}\psi_l^{m+1} - \dfrac{1}{v_{l-1}}\phi_{l-2} - \dfrac{\hat{u}_{l-1}}{v_{l-1}}\phi_{l-1} & 2 < l \leq L. \end{cases} \tag{126}
$$

The marginal PDF $p_{\mathbf{\Psi}}(\mathbf{\Psi})$ can be solved by integrating across $\mathbf{u}$:

$$
p_{\mathbf{\Psi}}(\mathbf{\Psi}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{\Psi u}}(\mathbf{\Psi}, \mathbf{u}) \quad d\mathbf{u}. \tag{127}
$$

Computational Complexity

The relations given in this document allows one to calculate the joint PDF of the pressure field envelope, $p_{\mathbf{\Psi}}(\mathbf{\Psi})$ at any range and depth, given a range independent environment and a depth varying, random sound speed profile $\mathbf{c}$ with PDF, $p_{\mathbf{c}}(\mathbf{c})$. The simplicity of the FD/FE approach and the resulting equations hide the computational complexity required in evaluation.

Consider the dimensionality of $p_{\mathbf{\Psi u}}(\mathbf{\Psi}, \mathbf{u})$. If one uses L mesh points, the PDF would have $2L-1$ dimensions. One objective of solving for $p_{\mathbf{\Psi}}(\mathbf{\Psi})$ is to calculate the its covariance matrix, $\Lambda_{\mathbf{\Psi}}$. The covariance matrix has $L(L-1)/2$ unique entries, each of which must be derived from a marginal joint PDF. One would need to integrate $p_{\mathbf{\Psi u}}(\mathbf{\Psi}, \mathbf{u})$ $2L-3$ times to yield the marginal joint PDF, and then integrate twice more to evaluate the covariance matrix entry. To fill $\Lambda_{\mathbf{\Psi}}$ would require $L(L-1)(2L-1)/2$ integrations. Assuming each integration across the PDF space requires evaluating $p_{\mathbf{\Psi u}}(\mathbf{\Psi}, \mathbf{u})$ at L different points, the total number of times these expressions would be evaluated would approach L^4. For an environment requiring 500 mesh points and a computational platform which can evaluate the PDF 100,000 times each second, this would amount to over 170 hours of CPU time.

Steps can be taken to reduce the overall complexity. For example, one could assume the PDF of ν were jointly Gaussian. The relationship between ν and $\mathbf{u}$ is linear, which would also be Gaussian. Unfortunately, Equations 106 through 109, which relate $\mathbf{\Psi}$ to $\mathbf{u}$ are nonlinear, since ψ and u are multiplied together in several terms. Thus the output ψ cannot explicitly be called Gaussian. As range (and m) increases, the Central Limit Theorem would likely take a role, making the final envelope statistics $p_{\mathbf{\Psi}}(\mathbf{\Psi})$ Gaussian in nature. Given current computational capabilities, it would be beneficial to investigate how the second moments of $\mathbf{\Psi}$ propagate through range, rather than its entire PDF.